A SHORT OBSTRUCTION TO UNIFORM PRIME-GAP ASYMPTOTICS AND THE ADDITIVE STRUCTURE OF THE ADMISSIBLE CONSTANTS

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Abstract. Let p_n be the n-th prime, let $g_n = p_{n+1} - p_n$, and put

$$d(x) = \max_{p_n < x} g_n.$$

For a fixed constant C, consider the uniform assertion

$$\pi(y + Cd(x)) - \pi(y) \sim \frac{Cd(x)}{\log y} \qquad (x/2 < y < x).$$

We give a compressed record-gap obstruction showing that two constants whose separation is less than 1 cannot both satisfy this assertion. We then prove that no $0 < C < 1$ can satisfy it by the same gap obstruction, while $C = 1$ is excluded by Westzynthius' theorem on large prime gaps. We then combine this with Tao's difference observation: if the assertion holds for C_1 and C_2, then it also holds for $C_1 - C_2$. Consequently the set of admissible constants is either trivial, or, under the natural signed convention, a cyclic lattice $C_0\mathbb{Z}$ for a unique $C_0 > 1$.

1. The uniform assertion

Let p_n denote the n-th prime and write

$$g_n = p_{n+1} - p_n.$$

For $x > 2$, define

$$d(x) = \max_{p_n < x}(p_{n+1} - p_n).$$

Thus a prime gap is counted as soon as its left endpoint is below x; its right endpoint need not be below x.

The motivating question is Erdős Problem #1138, as listed by Bloom [2]; its original source is [1, 1.3].

For fixed $C \neq 0$, let $A(C)$ denote the assertion

$$\sup_{x/2<y<x} \left| \frac{\pi(y + Cd(x)) - \pi(y)}{Cd(x)/\log y} - 1 \right| \longrightarrow 0 \qquad (x \to \infty). \tag{1}$$

For the structural statement, put

$$\mathcal{C} = \{0\} \cup \{C \neq 0 : A(C) \text{ holds}\},$$

with 0 adjoined only as the additive identity.

We shall use three standard inputs. First, prime gaps are unbounded, so strict record prime gaps occur infinitely often. Second, by the prime number theorem,

$$d(x) = o(x). \tag{2}$$

Indeed $p_{n+1}/p_n \to 1$, so every prime gap whose left endpoint is below x is $o(x)$. Third, we shall use Westzynthius' theorem, namely

$$\limsup_{n\to\infty} \frac{p_{n+1}-p_n}{\log p_n} = \infty. \tag{3}$$

Equivalently, maximal prime gaps are not asymptotic to the average prime gap size $\log p$; in fact their normalized limsup is infinite [3].

2. The compressed record-gap obstruction

Theorem 2.1. *Let* $1 < C_1 < C_2$ *and suppose*

$$0 < C_2 - C_1 < 1.$$

Then $A(C_1)$ *and* $A(C_2)$ *cannot both hold.*

Proof. Choose strict record prime gaps

$$(P_k, Q_k) = (p_{n_k}, p_{n_k+1}), \qquad D_k = Q_k - P_k,$$

so that $D_k > g_m$ for all $m < n_k$. Put

$$x_k = P_k + 1.$$

For all large k, we have $P_k < x_k < Q_k$, and since (P_k, Q_k) is a strict record gap,

$$d(x_k) = D_k.$$

Let

$$\eta = C_2 - C_1 \in (0,1), \qquad y_k = P_k - C_1 D_k.$$

By (2), we have $D_k = o(P_k)$, so for all large k,

$$x_k/2 < y_k < x_k.$$

Moreover

$$y_k + C_1 D_k = P_k, \qquad y_k + C_2 D_k = P_k + \eta D_k < Q_k.$$

Since there are no primes in (P_k, Q_k),

$$\pi(y_k + C_1 D_k) = \pi(P_k) = \pi(P_k + \eta D_k) = \pi(y_k + C_2 D_k).$$

Hence, setting

$$N_k = \pi(y_k + C_1 D_k) - \pi(y_k) = \pi(y_k + C_2 D_k) - \pi(y_k),$$

the two asserted asymptotics would give

$$N_k \sim \frac{C_1 D_k}{\log y_k} \qquad \text{and} \qquad N_k \sim \frac{C_2 D_k}{\log y_k}.$$

Thus $C_1 = C_2$, a contradiction. □

Corollary 2.2. *The assertion* $A(C)$ *cannot hold for every fixed* $C > 1$.

3. No constants in $(0, 1]$

Theorem 3.1. *No constant* $C \in (0, 1]$ *satisfies* $A(C)$*.*

Proof. Again let (P_k, Q_k) be strict record prime gaps and set

$$D_k = Q_k - P_k.$$

Choose, for instance,

$$x_k = P_k + \frac{1}{2} D_k.$$

Then $P_k < x_k < Q_k$, so $d(x_k) = D_k$. All starting points below are admissible for all large k, since $D_k = o(P_k)$.

First suppose $0 < C < 1$. Put

$$y_k = P_k + \frac{1 - C}{2} D_k.$$

Then both y_k and

$$y_k + CD_k = P_k + \frac{1 + C}{2} D_k$$

lie strictly inside the prime-free interval (P_k, Q_k). Hence

$$\pi(y_k + CD_k) - \pi(y_k) = 0,$$

whereas $A(C)$ would require the normalized ratio to tend to 1. Thus $A(C)$ fails for every $0 < C < 1$.

It remains to exclude $C = 1$. By Westzynthius' theorem, the record gaps may be chosen so that

$$\limsup_{k \to \infty} \frac{D_k}{\log P_k} = \infty.$$

Indeed, if some gap g_n has large ratio $g_n / \log p_n$, then the record gap up to that point has length at least g_n and left endpoint at most p_n, so its ratio is at least as large.

Now take

$$x_k = P_k + 1, \qquad y_k = P_k.$$

For all large k, $P_k < x_k < Q_k$, so again $d(x_k) = D_k$, and y_k is admissible. Since $Q_k = P_k + D_k$ is the next prime after P_k,

$$\pi(P_k + D_k) - \pi(P_k) = \pi(Q_k) - \pi(P_k) = 1.$$

Thus $A(1)$ would force

$$1 \sim \frac{D_k}{\log P_k}$$

along all record gaps, contradicting the preceding limsup. Hence $A(1)$ fails. □

4. The difference observation

The following observation is due to Terence Tao.

Lemma 4.1 (Difference closure)**.** *Suppose* $0 < C_2 < C_1$. *If* $A(C_1)$ *and* $A(C_2)$ *hold, then* $A(C_1 - C_2)$ *holds. Consequently, the positive admissible constants are closed under positive differences. Under the natural signed convention for (1),* $\mathcal{C}$ *is closed under subtraction.*

Proof. Put

$$\delta = C_1 - C_2, \qquad d = d(x).$$

Let $x/2 < y < x$. Since $d = o(x)$, all shifts of y by a fixed multiple of d have logarithm $(1 + o(1)) \log y$, uniformly in $x/2 < y < x$.

If $y + \delta d < x$, then both y and $y + \delta d$ are admissible starting points. Hence

$$\begin{aligned}\pi(y + \delta d) - \pi(y) &= [\pi(y + C_1 d) - \pi(y)] - [\pi(y + C_1 d) - \pi(y + \delta d)]\\ &\sim \frac{C_1 d}{\log y} - \frac{C_2 d}{\log(y + \delta d)} \sim \frac{\delta d}{\log y}.\end{aligned}$$

If $y + \delta d \geq x$, put

$$z = y - C_2 d.$$

Then for all large x,

$$x/2 < z < x.$$

Indeed $y \geq x - \delta d$, so $z \geq x - C_1 d > x/2$. Therefore

$$\begin{aligned}\pi(y + \delta d) - \pi(y) &= [\pi(z + C_1 d) - \pi(z)] - [\pi(z + C_2 d) - \pi(z)]\\ &\sim \frac{C_1 d}{\log z} - \frac{C_2 d}{\log z} \sim \frac{\delta d}{\log y}.\end{aligned}$$

The estimates are uniform in y, so $A(\delta)$ follows. □

Corollary 4.2. *No two distinct positive admissible constants can differ by at most* 1.

Proof. A difference in $(0, 1)$ is excluded by Theorem 2.1. A difference equal to 1 is excluded by Lemma 4.1 together with Theorem 3.1. □

5. Structure of the set of constants

Theorem 5.1. *Under the natural signed convention for* $A(C)$*, either*

$$\mathcal{C} = \{0\},$$

or there is a unique $C_0 > 1$ *such that*

$$\mathcal{C} = C_0\mathbb{Z} = \{C_0 k : k \in \mathbb{Z}\}.$$

Equivalently, the positive admissible constants are either empty or exactly

$$\{C_0, 2C_0, 3C_0, \ldots\}.$$

Proof. By Lemma 4.1, $\mathcal{C}$ is an additive subgroup of $\mathbb{R}$ under the signed convention. Let

$$\mathcal{C}_+ = \mathcal{C} \cap (0, \infty).$$

By Theorem 3.1,

$$\mathcal{C}_+ \cap (0, 1] = \emptyset.$$

If $\mathcal{C}_+ = \emptyset$, then $\mathcal{C} = \{0\}$. Otherwise set

$$C_0 = \inf \mathcal{C}_+.$$

Then $C_0 \geq 1$.

We first show that $C_0 \in \mathcal{C}_+$. If not, then there are distinct $a, b \in \mathcal{C}_+$ with

$$C_0 < a < b < C_0 + 1.$$

Since $\mathcal{C}$ is closed under subtraction, $b - a \in \mathcal{C}_+$. But $0 < b - a < 1$, contradicting Theorem 3.1. Hence $C_0 \in \mathcal{C}_+$. In particular $C_0 > 1$.

Now let $C \in \mathcal{C}_+$. Write

$$C = mC_0 + r, \qquad m \in \mathbb{Z}_{\geq 0}, \qquad 0 \leq r < C_0.$$

Since $\mathcal{C}$ is a subgroup and $C, C_0 \in \mathcal{C}$, the remainder $r = C - mC_0$ lies in $\mathcal{C}$. By the minimality of C_0, this forces $r = 0$. Thus every positive element of $\mathcal{C}$ is an integer multiple of C_0. Conversely, because $\mathcal{C}$ is a subgroup and $C_0 \in \mathcal{C}$, every integer multiple of C_0 lies in $\mathcal{C}$. Therefore $\mathcal{C} = C_0\mathbb{Z}$. The generator is unique because it is the least positive element of $\mathcal{C}$. ☐

6. Conclusion

The obstruction is caused by the definition of $d(x)$: a record prime gap is counted as soon as its left endpoint is below x, even if the gap extends far past x. Placing x just inside such a gap and taking $y = P - C_1 D$ makes the two endpoints

$$y + C_1 D = P, \qquad y + C_2 D = P + (C_2 - C_1)D$$

land in the same prime-free gap whenever $0 < C_2 - C_1 < 1$. The two prime counts are then identical, but the predicted main terms differ by the factors C_1 and C_2. Together with the exclusion of $(0, 1]$ and Tao's difference observation, this forces the admissible constants to be either trivial or a single arithmetic lattice.

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REFERENCES

[1] Various, *Some of Paul's favorite problems*. Booklet produced for the conference "Paul Erdős and his mathematics", Budapest, July 1999 (1999).

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[3] E. Westzynthius, Über die Verteilung der Zahlen, die zu den n ersten Primzahlen teilerfremd sind, *Commentationes Physico-Mathematicae* **5** (1931), 1–37.