

# Equivalence of Induction Principles

Well-Ordering, Strong Induction, and Ordinary Induction

Throughout, let

$$\mathbb{N} = \{1, 2, 3, \dots\}.$$

**Theorem 1.** The Well-Ordering Principle and Strong Induction are equivalent.

*Proof.* We prove both implications.

**Well-Ordering Principle  $\implies$  Strong Induction.** Assume the Well-Ordering Principle: every nonempty subset of  $\mathbb{N}$  has a least element.

Let  $P(n)$  be a statement such that

1.  $P(1)$  is true; and
2. for each  $k \in \mathbb{N}$ , if  $P(1), P(2), \dots, P(k)$  are all true, then  $P(k + 1)$  is true.

We show that  $P(n)$  is true for every  $n \in \mathbb{N}$ .

Suppose, for contradiction, that this conclusion is false. Then

$$S = \{n \in \mathbb{N} : P(n) \text{ is false}\}$$

is nonempty. By the Well-Ordering Principle,  $S$  has a least element; call it  $m$ . Because  $P(1)$  is true,  $m \neq 1$ , and hence  $m > 1$ .

The minimality of  $m$  implies that  $P(j)$  is true for every  $j < m$ . In particular,

$$P(1), P(2), \dots, P(m - 1)$$

are all true. Applying the inductive hypothesis with  $k = m - 1$  gives  $P(m)$ . This contradicts  $m \in S$ . Therefore  $S$  is empty, so  $P(n)$  is true for every  $n \in \mathbb{N}$ . Thus Strong Induction holds.

**Strong Induction  $\implies$  Well-Ordering Principle.** Assume Strong Induction. Let  $A \subseteq \mathbb{N}$  be nonempty. For each  $n \in \mathbb{N}$ , define

$$Q(n) : \quad n \in A \implies A \text{ has a least element.}$$

We prove  $Q(n)$  for every  $n$  by Strong Induction.

*Base case.* If  $1 \in A$ , then 1 is the least element of  $A$ . Thus  $Q(1)$  is true.

*Inductive step.* Assume  $Q(1), Q(2), \dots, Q(k)$  are true, and suppose  $k + 1 \in A$ . There are two cases:

1. If some  $t \in A$  satisfies  $t \leq k$ , then  $Q(t)$  is true by the inductive hypothesis. Since  $t \in A$ , it follows that  $A$  has a least element.
2. If no element of  $A$  is at most  $k$ , then every element of  $A$  is at least  $k + 1$ . Since  $k + 1 \in A$ , the number  $k + 1$  is the least element of  $A$ .

In either case,  $k + 1 \in A$  implies that  $A$  has a least element. Therefore  $Q(k + 1)$  is true.

By Strong Induction,  $Q(n)$  is true for every  $n \in \mathbb{N}$ . Since  $A$  is nonempty, choose  $a \in A$ . Then  $Q(a)$  implies that  $A$  has a least element. Hence every nonempty subset of  $\mathbb{N}$  has a least element, so the Well-Ordering Principle holds.  $\square$

**Theorem 2.** Ordinary Induction and Strong Induction are equivalent.

*Proof.* Again, we prove both implications.

**Ordinary Induction  $\implies$  Strong Induction.** Assume Ordinary Induction. Suppose  $P(n)$  satisfies the hypotheses of Strong Induction:

1.  $P(1)$  is true; and
2. for each  $k \in \mathbb{N}$ , if  $P(1), P(2), \dots, P(k)$  are all true, then  $P(k + 1)$  is true.

Define

$$R(n) := P(1) \wedge P(2) \wedge \dots \wedge P(n).$$

We prove  $R(n)$  for every  $n$  by Ordinary Induction.

*Base case.* The statement  $R(1)$  is exactly  $P(1)$ , so it is true.

*Inductive step.* Assume  $R(k)$  is true. Then  $P(1), P(2), \dots, P(k)$  are all true, so the strong inductive hypothesis gives  $P(k + 1)$ . Consequently,

$$R(k + 1) = P(1) \wedge \dots \wedge P(k) \wedge P(k + 1)$$

is true.

By Ordinary Induction,  $R(n)$  is true for every  $n \in \mathbb{N}$ . Because  $R(n)$  includes  $P(n)$ , it follows that  $P(n)$  is true for every  $n \in \mathbb{N}$ . Thus Strong Induction holds.

**Strong Induction  $\implies$  Ordinary Induction.** Assume Strong Induction. Suppose  $P(n)$  satisfies the hypotheses of Ordinary Induction:

1.  $P(1)$  is true; and
2. for each  $k \in \mathbb{N}$ ,  $P(k) \implies P(k + 1)$ .

To verify the strong inductive step, fix  $k \in \mathbb{N}$  and assume that  $P(1), P(2), \dots, P(k)$  are all true. In particular,  $P(k)$  is true. The ordinary inductive hypothesis therefore gives  $P(k + 1)$ .

Thus the hypotheses of Strong Induction are satisfied. It follows that  $P(n)$  is true for every  $n \in \mathbb{N}$ , which is precisely the conclusion of Ordinary Induction.  $\square$