Gauss–Bonnet as "Curvature Counts Turning"

A moving-frame and triangulation proof

Let M be a compact oriented Riemannian surface whose boundary is piecewise C^2. Orient ∂M so that M stays on the left. If K is Gaussian curvature, k_g is signed geodesic curvature, and $\alpha_1, \ldots, \alpha_r$ are the signed exterior turning angles at the corners, then

$$\int_M K \, \mathrm{d}A + \int_{\partial M} k_g \, \mathrm{d}s + \sum_{j=1}^{r} \alpha_j = 2\pi\chi(M).$$

The three terms on the left are three forms of the same phenomenon: rotation of a direction. The boundary tangent can turn continuously (k_g), jump at a corner (α_j), or acquire rotation because parallel transport through the interior is obstructed by curvature (K). Gauss–Bonnet says that their total is forced by topology.

1. The local turning formula

Let $D \subset M$ be an oriented topological disk on which there is a smooth positively oriented orthonormal frame (e_1, e_2). Its connection one-form ω is defined by

$$\nabla_X e_1 = \omega(X) e_2, \qquad \nabla_X e_2 = -\omega(X) e_1.$$

The structure equation for a surface is

$$d\omega = -K \, \mathrm{d}A. \tag{1}$$

Along a smooth boundary arc, write its positively oriented unit tangent as

$$T = \cos\theta \, e_1 + \sin\theta \, e_2.$$

Differentiating covariantly with respect to arclength s gives

$$\nabla_T T = \big(\theta'(s) + \omega(T)\big)\big(-\sin\theta \, e_1 + \cos\theta \, e_2\big).$$

The vector in the second pair of parentheses is the left unit normal to the boundary. Hence

$$k_g = \theta'(s) + \omega(T). \tag{2}$$

Now traverse ∂D once. The angle θ changes smoothly along each arc and jumps by the exterior angle α_j at each corner. Since the tangent makes one full positive turn around a disk,

$$\sum_{\text{arcs}} \int d\theta + \sum_j \alpha_j = 2\pi. \tag{3}$$

Integrating (2), adding the corner jumps, and using (3), Stokes' theorem, and (1), we obtain

$$\begin{aligned}\int_{\partial D} k_g \, \mathrm{d}s + \sum_j \alpha_j &= 2\pi + \int_{\partial D} \omega \\ &= 2\pi + \int_D d\omega \\ &= 2\pi - \int_D K \, \mathrm{d}A.\end{aligned}$$

Therefore

$$\boxed{\int_D K \, \mathrm{d}A + \int_{\partial D} k_g \, \mathrm{d}s + \sum_j \alpha_j = 2\pi.} \tag{4}$$

Geometric interpretation. Equation (2) separates the observed turning of the tangent into turning relative to a chosen frame and turning of the frame itself. Equation (1) says that the infinitesimal failure of the frame to return unchanged is precisely Gaussian curvature. Thus (4) is literally a conservation law for turning.

2. From disks to an arbitrary compact surface

Choose a sufficiently fine triangulation of M, with boundary vertices at every corner of ∂M. Apply the disk formula (4) to every triangle and sum.

- The curvature integrals add to $\int_M K \, \mathrm{d}A$.
- Every interior edge occurs twice with opposite orientations, so its two geodesic-curvature integrals cancel.
- The uncancelled edge integrals are exactly $\int_{\partial M} k_g \, \mathrm{d}s$.
- At an interior vertex, the angles from all incident triangles fill a complete circle. Their total local contribution is 2π. At a boundary vertex, the same bookkeeping leaves the exterior turn of the actual boundary.

Here is the angle bookkeeping explicitly. Let V_i, V_b denote the numbers of interior and boundary vertices, and let E_i, E_b, F denote the numbers of interior edges, boundary edges, and triangles. If β is a triangle's interior angle at a vertex, its exterior angle is $\pi - \beta$. The triangle angles around an interior vertex sum to 2π; those around a boundary vertex v sum to the interior angle ϕ_v of M. Summing (4) over the F triangles therefore gives

$$\int_M K \, \mathrm{d}A + \int_{\partial M} k_g \, \mathrm{d}s + 3\pi F - 2\pi V_i - \sum_{v \in \partial M} \phi_v = 2\pi F.$$

Since the boundary exterior angle is $\alpha_v = \pi - \phi_v$, this becomes

$$\int_M K \, \mathrm{d}A + \int_{\partial M} k_g \, \mathrm{d}s + \sum_v \alpha_v = 2\pi V_i + \pi V_b - \pi F. \tag{5}$$

Every boundary component is a polygon, so $E_b = V_b$, while counting triangle-edge incidences gives $3F = 2E_i + E_b$. Consequently,

$$\begin{aligned} 2\pi\chi(M) &= 2\pi(V_i + V_b - E_i - E_b + F) \\ &= 2\pi V_i + \pi V_b - \pi F. \end{aligned}$$

Comparing this with (5) proves the theorem.

3. The closed-surface case in one line of counting

When $\partial M = \varnothing$, take a geodesic triangulation. For a triangle with interior angles $\beta_1, \beta_2, \beta_3$, (4) becomes

$$\int_{\triangle} K \, \mathrm{d}A + (\pi - \beta_1) + (\pi - \beta_2) + (\pi - \beta_3) = 2\pi,$$

or

$$\int_{\triangle} K \, \mathrm{d}A = \beta_1 + \beta_2 + \beta_3 - \pi. \tag{6}$$

Thus curvature is the excess of the angle sum over the Euclidean value π. Summing (6) over all triangles gives

$$\int_M K \, \mathrm{d}A = 2\pi V - \pi F.$$

Since every edge belongs to two triangles, $3F = 2E$, and hence

$$\int_M K \, \mathrm{d}A = 2\pi \left(V - \frac{F}{2} \right) = 2\pi(V - E + F) = 2\pi\chi(M).$$