The Fundamental Theorem of Algebra

A proof via Liouville's theorem

Fundamental Theorem of Algebra. Every nonconstant polynomial $p \in \mathbb{C}[z]$ has at least one complex root.

Liouville's Theorem. *Every bounded entire function $f : \mathbb{C} \to \mathbb{C}$ is constant.*

Proof

Let

$$p(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_0, \qquad a_n \neq 0, \quad n \geq 1.$$

Suppose, toward a contradiction, that p has no zeros in $\mathbb{C}$. Then

$$f(z) = \frac{1}{p(z)}$$

is entire.

We claim that f is bounded. First, factor out the leading term:

$$p(z) = a_n z^n \left(1 + \frac{a_{n-1}}{a_n z} + \frac{a_{n-2}}{a_n z^2} + \cdots + \frac{a_0}{a_n z^n}\right).$$

The expression in parentheses tends to 1 as $|z| \to \infty$. Therefore there exists $R > 0$ such that, whenever $|z| \geq R$,

$$\left|1 + \frac{a_{n-1}}{a_n z} + \cdots + \frac{a_0}{a_n z^n}\right| \geq \frac{1}{2}.$$

Consequently,

$$|p(z)| \geq \frac{|a_n|}{2}|z|^n \quad \text{and hence} \quad |f(z)| \leq \frac{2}{|a_n||z|^n} \leq \frac{2}{|a_n|R^n} \qquad (|z| \geq R). \tag{1}$$

Thus f is bounded outside the disk $\{z : |z| \leq R\}$.

Inside that disk, f is continuous on a compact set. It therefore attains a finite maximum:

$$M = \max_{|z| \leq R} |f(z)| < \infty. \tag{2}$$

Equations (1) and (2) show that f is bounded on all of $\mathbb{C}$. By Liouville's theorem, f is constant. Since $f = 1/p$, the polynomial p must also be constant, contradicting $n \geq 1$.

Therefore p has a complex root. □